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Periodic solutions of autonomous Newtonian systems partially asymptotically linear at infinity

Justyna Fura

Faculty of Mathematics and Computer Science, Nicolaus Copernicus University, PL-87-100 Toruń, ul. Chopina 12/18, Poland

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ABSTRACT

The goal of this article is to study the existence, continuation and global bifurcation of autonomous Newtonian systems partially asymptotically linear at infinity. The main idea is to prove a kind of splitting lemma and apply the degree for S^1-equivariant gradient maps defined by Rybicki (1994) in [16].

1. Introduction

In this paper we study the existence of nonstationary 2π-periodic solutions of the autonomous Newtonian systems

$$\ddot{x}(t) = -V'(x(t)) \tag{1.1}$$

where $V \in C^2(\mathbb{R}^n, \mathbb{R})$ and V' denotes the gradient of V. We assume that V' is partially asymptotically linear at infinity, i.e.

$$V'_{x_1}(x) = Bx_1 + o(|x_1|) \quad \text{as } |x_1| \to \infty, \tag{1.2}$$

where $x = (x_1, x_2) \in \mathbb{R}^{n_1} \oplus \mathbb{R}^{n_2} = \mathbb{R}^n$ and B is $(n_1 \times n_1)$-symmetric matrix. Moreover, we assume that $(V')^{-1}(0) = \{p_1, \ldots, p_q\}$ is finite.

Asymptotically linear systems, i.e. systems which are close at infinity to linear systems, have been studied by many authors, see [1–12]. The Newtonian system (1.1) is asymptotically linear, if the right hand site of equation satisfies the following condition:

$$V'_x(x) = Ax + o(|x|) \quad \text{as } |x| \to \infty, \tag{1.3}$$

where $x \in \mathbb{R}^n$ and A is $(n \times n)$-symmetric matrix. This condition implies, that gradient of functional corresponding to the system (1.1) is close at infinity to linear operator and for such operators we have computational formulas of different topological invariants. Therefore, if we study system (1.1) via variational and topological methods, then the condition (1.3) is a very natural assumption. The motivation to consider partially asymptotically linear Newtonian systems is paper [13]. Jiang in [13] considered system (1.1) with partially superquadratic potential V and has shown bifurcation of nonstationary periodic solutions of (1.1). Autonomous Newtonian systems (analogously Hamiltonian systems) with potential satisfying superquadratic condition have been intensively studied, see [14,15,12] for instance. This is one of the typical assumptions on the potential to prove the existence of sequence of nonstationary periodic solutions. Jiang has weakened this assumption

E-mail address: Justyna.Fura@mat.umk.pl.

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and considered system (1.1) with potential V which is close at infinity to $\frac{1}{2}(Bx_1, x_1) + V_2(x_2)$, where $x = (x_1, x_2) \in \mathbb{R}^n = \mathbb{R}^{n_1} \oplus \mathbb{R}^{n_2}$, B is $(n_1 \times n_1)$-symmetric matrix and V_2 satisfies some superquadratic condition. Analogously, since the assumption of asymptotically linearity of gradient V' is natural when we apply variational and topological methods, we wanted to weaken this crucial assumption. Therefore, we study Newtonian system (1.1) with potential V satisfying condition (1.2) instead of condition (1.3) and we prove the existence, continuation and bifurcation results.

Since system (1.1) has natural S^1-symmetry given by shift in time, we use as a topological tool the degree for S^1-equivariant gradient maps, see [16]. We treat solutions of (1.1) as critical points of S^1-invariant functional defined on the Sobolev space $\mathbb{H}^1_{2\pi}$, see also [10,12]. Using the homotopy introduced by Jiang in [13], we prove the splitting lemma, and compute the index at infinity in terms of the degree for S^1-equivariant gradient maps. In [17] we have computed indices at stationary solutions $p_1, \dots, p_q$ in both non-resonant and resonant cases. Combining these results, we proved the existence of at least one nonstationary 2π-periodic solution of (1.1).

As a consequence of the existence result we have obtained the continuation of nonstationary 2π-periodic solutions of the following system:

$$\begin{cases} \ddot{x}(t) = -V'(x(t), \lambda), \\ x(0) = x(2\pi), \\ \dot{x}(0) = \dot{x}(2\pi), \end{cases} \tag{1.4}$$

where $V \in C^2(\mathbb{R}^n \times \mathbb{R}, \mathbb{R})$ and potential $V_0 := V(\cdot, 0)$ possess all the properties of potential V in (1.1). We formulate sufficient conditions for the existence of connected sets of 2π-periodic solutions of (1.4) emanating from level $\lambda = 0$.

Finally, we study system (1.4) for the family of potentials $V \in C^2(\mathbb{R}^n \times \mathbb{R}, \mathbb{R})$ whose gradients $V'_x(\lambda_\pm, \cdot)$ are partially asymptotically linear at infinity for some $\lambda_- < \lambda_+$. We formulate sufficient conditions for the existence of closed connected sets of nonstationary 2π-periodic solutions of system (1.4) emanating from infinity. We prove it by applying variational methods and the abstract result from the paper [18], see Theorem 2.2. We also consider system (1.4) with $V \in C^2(\mathbb{R}^n \times \mathbb{R}, \mathbb{R})$ such that the gradient V'_x is partially asymptotically linear at infinity, i.e. $V'_{x_1}(x, \lambda) = B(\lambda)x_1 + o(\|x_1\|)$ as $\|x_1\| \to \infty$ uniformly on compact subsets of $\mathbb{R}^{n_2}$ and bounded λ-intervals and $B(\lambda)$ is a real symmetric matrix for every $\lambda \in \mathbb{R}$. Under this assumption we prove additional properties of obtained continuum of periodic solutions of system (1.4).

It is worth pointing out that since we study connected sets of periodic solutions we cannot use the Conley index and the Morse theory, see [19–23] for discussion and examples. Moreover, we also show that one cannot prove the results of this paper using the Leray–Schauder degree, because, in the presence of S^1-symmetry, it vanishes, see [24,7,11].

After introduction this article is organized in the following way.

In Section 2 we summarize several basic notations and facts from the degree theory for S^1-equivariant gradient maps. We finish this section with theorems concerning the existence of connected sets of critical orbits of S^1-invariant functionals and Dancer result concerning computation of the Leray–Schauder degree for S^1-equivariant operators.

The main result of Section 3 is Lemma 3.2, in which using a kind of splitting lemma, see Lemma 3.1, we compute index at infinity for partially asymptotically linear system (1.1) in terms of the degree for S^1-equivariant gradient maps.

In Section 4 we formulate the main results of this article. We prove the sufficient conditions for the existence of nonstationary 2π-periodic solutions of partially asymptotically linear system (3.1), see Theorems 4.1, 4.2. We would like to emphasize that we cannot use in our proofs the Leray–Schauder degree, see Remark 4.2. As a consequence of existence theorem we obtain the continuation of nonstationary 2π-periodic solutions of family of nonlinear equations (4.2), see Theorem 4.3. We also study bifurcation from infinity of periodic solutions of family (4.2). In Theorem 4.4 we have formulated sufficient conditions for the existence of unbounded closed connected sets of 2π-periodic solutions of system (4.2). In Theorem 4.5 we have proved additional properties of obtained component.

In Section 5 we illustrate results proved in Section 4.

2. Preliminaries

In this section, for the convenience of the reader, we collect several basic facts from the degree theory for S^1-equivariant gradient maps defined in [16]. For a treatment of a general theory of the degree for G-equivariant maps, where G is an arbitrary compact Lie group, see [25]. The degree for S^1-equivariant gradient maps will be denoted briefly by ∇_{S^1}-deg. The sketch of the definition and properties of this degree one can find in [18].

Put $U(S^1) = \mathbb{Z} \oplus \bigoplus_{k=1}^{\infty} \mathbb{Z}$ and define the actions

$$+, \star : U(S^1) \times U(S^1) \to U(S^1),$$

as follows

$$\alpha + \beta = (\alpha_0 + \beta_0, \alpha_1 + \beta_1, \dots, \alpha_k + \beta_k, \dots), \tag{2.1}$$

$$\alpha \star \beta = (\alpha_0\beta_0, \alpha_0\beta_1 + \beta_0\alpha_1, \dots, \alpha_0\beta_k + \beta_0\alpha_k, \dots), \tag{2.2}$$

where $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_k, \dots), \beta = (\beta_0, \beta_1, \dots, \beta_k, \dots) \in U(S^1)$. It is easy to check that $(U(S^1), +, \star)$ is a commutative ring with the trivial element $\Theta = (0, 0, \dots) \in U(S^1)$ and the unit $\mathbb{I} = (1, 0, \dots) \in U(S^1)$. The ring $(U(S^1), +, \star)$ is called the Euler ring of the group S^1. For the definition of the Euler ring $U(G)$, where G is any compact Lie group, we refer the reader to [26].

Let $\mathbb{V}$ be a finite dimensional, orthogonal S^1-representation. Fix $k \in \mathbb{N}$ and set $C^k_{S^1}(\mathbb{V}, \mathbb{R}) = \{f \in C^k(\mathbb{V}, \mathbb{R}) : f \text{ is } S^1\text{-invariant}\}$. Let $f_0 \in C^1_{S^1}(\mathbb{V}, \mathbb{R})$. Choose an open, bounded and S^1-invariant subset $\Omega \subset \mathbb{V}$ such that $(\nabla f_0)^{-1}(0) \cap \partial\Omega = \emptyset$. Under these assumptions the degree for S^1-equivariant gradient maps $\nabla_{S^1}\text{-deg}(\nabla f_0, \Omega) \in U(S^1)$ with coordinates

$$\nabla_{S^1}\text{-deg}(\nabla f_0, \Omega) = (\nabla_{S^1}\text{-deg}_{S^1}(\nabla f_0, \Omega), \nabla_{S^1}\text{-deg}_{\mathbb{Z}_1}(\nabla f_0, \Omega), \ldots, \nabla_{S^1}\text{-deg}_{\mathbb{Z}_k}(\nabla f_0, \Omega), \ldots)$$

is well defined, see [16,18]. This degree has all natural properties demanded from every degree theory, see Theorem 2.1 in [18].

To apply successfully any degree theory we need computational formulas for this invariant. Below we will present how to compute the degree for S^1-equivariant gradient maps of a linear, self-adjoint, S^1-equivariant isomorphism.

For $k \in \mathbb{N}$ define a map $\rho^k : S^1 \to GL(2, \mathbb{R})$ as follows

$$\rho^k(e^{i\theta}) = \begin{bmatrix} \cos(k\theta) & -\sin(k\theta) \\ \sin(k\theta) & \cos(k\theta) \end{bmatrix} \quad 0 \le \theta < 2\pi.$$

For $j, k \in \mathbb{N}$ we denote by $\mathbb{R}[j, k]$ the direct sum of j copies of $(\mathbb{R}^2, \rho^k)$, we also denote by $\mathbb{R}[j, 0]$ the trivial j-dimensional S^1-representation. We say that two S^1-representations $\mathbb{V}$ and $\mathbb{W}$ are equivalent (we denote it $\mathbb{V} \approx \mathbb{W}$), if there exists an S^1-equivariant, linear isomorphism $T : \mathbb{V} \to \mathbb{W}$.

We will denote by $m^-(L)$ the Morse index of a symmetric matrix L i.e. the sum of algebraic multiplicities of negative eigenvalues of L.

Lemma 2.1 ([16]). *If $\mathbb{V} \approx \mathbb{R}[j_0, 0] \oplus \mathbb{R}[j_1, k_1] \oplus \cdots \oplus \mathbb{R}[j_r, k_r]$, $L : \mathbb{V} \to \mathbb{V}$ is a self-adjoint, S^1-equivariant, linear isomorphism and $\gamma > 0$ then*

(1) $L = \text{diag}\,(L_0, L_1, \ldots, L_r)$,
(2)

$$\nabla_{S^1}\text{-deg}_H(L, B_\gamma(\mathbb{V})) = \begin{cases} (-1)^{m^-(L_0)}, & \text{for } H = S^1, \\ (-1)^{m^-(L_0)} \cdot \dfrac{m^-(L_i)}{2}, & \text{for } H = \mathbb{Z}_{k_i} \\ 0, & \text{for } H \notin \{S^1, \mathbb{Z}_{k_1}, \ldots, \mathbb{Z}_{k_r}\}, \end{cases}$$

(3) in particular, if $L = -Id$, then

$$\nabla_{S^1}\text{-deg}_H(-Id, B_\gamma(\mathbb{V})) = \begin{cases} (-1)^{j_0}, & \text{for } H = S^1, \\ (-1)^{j_0} \cdot j_i, & \text{for } H = \mathbb{Z}_{k_i}, \\ 0, & \text{for } H \notin \{S^1, \mathbb{Z}_{k_1}, \ldots, \mathbb{Z}_{k_r}\}. \end{cases}$$

Let $(\mathbb{H}, \langle\cdot,\cdot\rangle_{\mathbb{H}})$ be an infinite-dimensional, separable Hilbert space which is an orthogonal S^1-representation and let $C^k_{S^1}(\mathbb{H}, \mathbb{R})$ denote the set of S^1-invariant C^k-functionals. Fix $\Phi \in C^1_{S^1}(\mathbb{H}, \mathbb{R})$ such that $\nabla\Phi(u) = u - \nabla\eta(u)$, where $\nabla\eta : \mathbb{H} \to \mathbb{H}$ is an S^1-equivariant compact operator. Let $\mathcal{U} \subset \mathbb{H}$ be an open, bounded and S^1-invariant set such that $(\nabla\Phi)^{-1}(0) \cap \partial\mathcal{U} = \emptyset$. In this situation $\nabla_{S^1}\text{-deg}(Id - \nabla\eta, \mathcal{U}) \in U(S^1)$ is well-defined, see [16] for details and basic properties of this degree.

Below we formulate the continuation theorem for S^1-equivariant gradient operators in the form of a compact perturbation of the identity. The proof of this theorem is standard.

Theorem 2.1. *Let $\Phi \in C^1_{S^1}(\mathbb{H} \times \mathbb{R}, \mathbb{R})$ be such that $\nabla_u\Phi(u, \lambda) = u - \nabla_u\eta(u, \lambda)$, where $\nabla\eta : \mathbb{H} \times \mathbb{R} \to \mathbb{H}$ is an S^1-equivariant compact operator. Fix an open, bounded and S^1-invariant subset $\mathcal{U} \subset \mathbb{H}$ and $\lambda_0 \in \mathbb{R}$ such that*

(1) $(\nabla_u\Phi(\cdot, \lambda_0))^{-1}(0) \cap \partial\mathcal{U} = \emptyset$,
(2) $\nabla_{S^1}\text{-deg}(\nabla_u\Phi(\cdot, \lambda_0), \mathcal{U}) \neq \Theta \in U(S^1)$.

Then there exists continua (closed connected sets) $\mathcal{C}^\pm \subset \mathbb{H} \times \mathbb{R}$, with

$$\mathcal{C}^- \subset ((-\infty, \lambda_0] \times \mathbb{H}) \cap (\nabla_u\Phi(\cdot, \lambda_0))^{-1}(0),$$
$$\mathcal{C}^+ \subset ([\lambda_0, +\infty) \times \mathbb{H}) \cap (\nabla_u\Phi(\cdot, \lambda_0))^{-1}(0),$$

and for both $\mathcal{C} = \mathcal{C}^\pm$ the following statements are valid

(1) $\mathcal{C} \cap (\{\lambda_0\} \times \mathcal{U}) \neq \emptyset$,
(2) *either $\mathcal{C}$ is unbounded or else $\mathcal{C} \cap (\mathbb{H} \setminus cl(\mathcal{U})) \neq \emptyset$.*

Now we present result proved in [18], which gives sufficient conditions for the existence of closed connected component of critical orbits of S^1-invariant C^2-functionals bifurcating from infinity.

Theorem 2.2 ([18]). *Let $(\mathbb{H}, \langle\cdot,\cdot\rangle_{\mathbb{H}})$ be an infinite-dimensional, separable Hilbert space which is an orthogonal S^1-representation and $\Phi \in C^2_{S^1}(\mathbb{H}\times\mathbb{R}, \mathbb{R})$ be functional, whose gradient is of the form compact perturbation of the identity. Let $\lambda_\pm \in \mathbb{R},\ \gamma > 0$ be such that*

$$(\nabla_u\Phi(\cdot,\lambda_\pm))^{-1}(0) \cap ((\mathbb{H}\setminus B_\gamma(\mathbb{H}))\times\{\lambda_\pm\}) = \emptyset. \tag{2.3}$$

If

$$\mathrm{BIF}(\infty,[\lambda_-,\lambda_+]) = \nabla_{S^1}\text{-deg}(\nabla_u\Phi(\cdot,\lambda_+), B_\gamma(\mathbb{H})) - \nabla_{S^1}\text{-deg}(\nabla_u\Phi(\cdot,\lambda_-), B_\gamma(\mathbb{H})) \neq \Theta,$$

then there exists an unbounded closed connected component C of

$$(\nabla_u\Phi)^{-1}(0) \cap (\mathbb{H}\times[\lambda_-,\lambda_+])$$

such that $C \cap (B_\gamma(\mathbb{H})\times\{\lambda_-,\lambda_+\}) \neq \emptyset$.

We finish this section with the theorem due to Dancer, see [24].

Theorem 2.3 ([24]). *Let $\mathcal{U} \subset \mathbb{H}$ be an open, bounded and S^1-invariant set and let $f \in C^0_{S^1}(cl(\mathcal{U}), \mathbb{H})$ be an operator in the form of a compact perturbation of the identity such that $0 \notin f(\partial\mathcal{U})$. Then $\deg_{LS}(f,\mathcal{U},0) = \deg_{LS}(f^{S^1}, \mathcal{U}^{S^1}, 0)$, where $\deg_{LS}$ denotes the Leray–Schauder degree.*

3. Index at infinity for partially asymptotically linear Newtonian systems

In this section we consider partially asymptotically linear Newtonian systems, i.e.

$$\begin{cases} \ddot{u}(t) = -V'(u(t)), \\ u(0) = u(2\pi), \\ \dot{u}(0) = \dot{u}(2\pi), \end{cases} \tag{3.1}$$

with potential V satisfying the following assumptions:

A.1 $V \in C^2(\mathbb{R}^n, \mathbb{R})$ is of the form

$$V(x) = V(x_1, x_2) = \frac{1}{2}(Bx_1, x_1) + V_2(x_2) + W(x),$$

where $x = (x_1, x_2) \in \mathbb{R}^{n_1} \oplus \mathbb{R}^{n_2} = \mathbb{R}^n$ and B is $(n_1 \times n_1)$-symmetric matrix,

A.2 $\sigma(B) \cap \{k^2 : k \in \mathbb{N}\cup\{0\}\} = \emptyset$,

A.3 $(-V_2'(x_2), x_2) \geq |x_2|^2$,

A.4 $W'_{x_1}(x) = o(|x_1|)$ as $|x_1| \to \infty$ uniformly for compact subset of $\mathbb{R}^{n_2}$, i.e.

$$\forall_{R_2>0}\forall_{\epsilon>0}\ \exists_{R_1>0}\ \forall_{x=(x_1,x_2)\in\mathbb{R}^n}\quad |x_2| \leq R_2\ \wedge |x_1| > R_1 \Rightarrow |W'_{x_1}(x)| < \epsilon|x_1|, \tag{3.2}$$

A.5 $\exists_{M>0}\ \ \forall_{x\in\mathbb{R}^n}\ \ (W'_{x_2}(x), x_2) < M.$

Consider S^1-invariant C^2-functional $\Phi_V : \mathbb{H}^1_{2\pi} \to \mathbb{R}$ given by the formula

$$\Phi_V(u) = \frac{1}{2}\int_0^{2\pi} |\dot{u}(t)|^2\, dt - \int_0^{2\pi} V(u(t))\, dt, \tag{3.3}$$

whose critical points are in one-to-one correspondence with weak solutions of (3.1). However, since the potential V is of C^2class, in fact, critical points of functional (3.3) are in one-to-one correspondence with solutions of (3.1). We will compute the degree for S^1-equivariant gradient maps of $\nabla\Phi_V$ on $B_R(\mathbb{H}^1_{2\pi})$ where $R > 0$ is sufficiently large radius. For this purpose we will construct S^1-invariant admissible gradient homotopy.

Let $\pi_i : \mathbb{R}^n \to \mathbb{R}^n$, for $i = 1, 2$ be an orthogonal projection on the subspace $\mathbb{R}^{n_i}$. Consider $u \in \mathbb{H}^1_{2\pi}$, $u : [0, 2\pi] \to \mathbb{R}^n$, $u(t) = (u_1(t), \ldots, c, u_n(t))$. Then the following composition are defined

$$\pi_1 \circ u : [0, 2\pi] \to \mathbb{R}^n, \quad (\pi_1 \circ u)(t) = (u_1(t), \ldots, u_{n_1}(t), 0, \ldots, 0)$$

and

$$\pi_2 \circ u : [0, 2\pi] \to \mathbb{R}^n, \quad (\pi_2 \circ u)(t) = (0, \ldots, 0, u_{n_1+1}(t), \ldots, u_{n_1+n_2}(t)).$$

Define $P : \mathbb{H}^1_{2\pi} \to \mathbb{H}^1_{2\pi}$ by the formula $P(u) = \pi_1 \circ u$ and note that $P^2 = P$, $\operatorname{im}(P) = \pi_1(\mathbb{H}^1_{2\pi}) = \mathbb{H}^1([0, 2\pi], \mathbb{R}^{n_1})$, $\ker(P) = \pi_2(\mathbb{H}^1_{2\pi}) = \mathbb{H}^1([0, 2\pi], \mathbb{R}^{n_2})$. Notice that $\ker(P)^\perp = \operatorname{im}(P)$ and P is an orthogonal projection. Denote $\mathcal{W}_1 := \operatorname{im}(P)$, $\mathcal{W}_2 := \ker(P)$. Obviously

$$\mathbb{H}^1_{2\pi} = \mathbb{H}^1([0, 2\pi], \mathbb{R}^n) = \mathbb{H}^1([0, 2\pi], \mathbb{R}^{n_1}) \oplus \mathbb{H}^1([0, 2\pi], \mathbb{R}^{n_2}) = \mathcal{W}_1 \oplus \mathcal{W}_2.$$

By $I - P : \mathbb{H}^1_{2\pi} \to \mathbb{H}^1_{2\pi}$ we will denote an orthogonal projection on subspace $\mathcal{W}_2$.

Define the following maps:

$$J_1 : \mathcal{W}_1 \to \mathbb{R}, \quad J_1(u_1) = \frac{1}{2} \int_0^{2\pi} |\dot{u}_1(t)|^2 - (Bu_1(t), u_1(t))\, dt, \tag{3.4}$$

$$J_2 : \mathcal{W}_2 \to \mathbb{R}, \quad J_2(u_2) = \int_0^{2\pi} \frac{1}{2} |\dot{u}_2(t)|^2 - V_2(u_2(t))\, dt, \tag{3.5}$$

$$\eta : \mathbb{H}^1_{2\pi} \to \mathbb{R}, \quad \eta(u) = \int_0^{2\pi} W(u(t))\, dt. \tag{3.6}$$

Notice that

- $\nabla J_1(u_1) = (Id - L)u_1$, where operator $L : \mathcal{W}_1 \to \mathcal{W}_1$ is defined by the formula

$$\langle Lu_1, v_1 \rangle = \int_0^{2\pi} (u_1(t) + Bu_1(t), v_1(t))\, dt, \tag{3.7}$$

- $\nabla J_2(u_2) = (Id - \nabla\xi)u_2$, where mapping $\nabla\xi : \mathcal{W}_2 \to \mathcal{W}_2$ is defined by the formula

$$\langle \nabla\xi(u_2), v_2 \rangle = \int_0^{2\pi} (u_2(t) - V_2'(u_2(t)), v_2(t))\, dt, \tag{3.8}$$

- $\nabla\eta : \mathbb{H}^1_{2\pi} \to \mathbb{H}^1_{2\pi}$ is defined by the formula

$$\langle \nabla\eta(u), v \rangle = \int_0^{2\pi} (W'(u(t)), v(t))\, dt. \tag{3.9}$$

It is well known that operators given by the formulas (3.6)–(3.8) are compact. Moreover assumption **A.2** and Corollary 5.1.1. in [17] implies that a linear operator $\nabla J_1(u_1) = (Id - L)u_1$ is an isomorphism. Thus the following linear system

$$\begin{cases} \ddot{u_1}(t) = -Bu_1(t), \\ u_1(0) = u_1(2\pi), \\ \dot{u_1}(0) = \dot{u_1}(2\pi), \end{cases} \tag{3.10}$$

has no nonzero periodic solutions.

Let $u \in \mathbb{H}^1_{2\pi} = \mathcal{W}_1 \oplus \mathcal{W}_2$. Then u has unique representation $u = (u_1, u_2)$, where $u_1 = P(u) \in \mathcal{W}_1$, $u_2 = (I - P)(u) \in \mathcal{W}_2$. Obviously $\|u_1\|_{\mathbb{H}^1_{2\pi}} = \|u_1\|_{\mathcal{W}_1}$ and $\|u_2\|_{\mathbb{H}^1_{2\pi}} = \|u_2\|_{\mathcal{W}_2}$, where norms $\|\cdot\|_{\mathcal{W}_1}$, $\|\cdot\|_{\mathcal{W}_2}$ are standard norms in the space $\mathcal{W}_1 = \mathbb{H}^1([0, 2\pi], \mathbb{R}^{n_1})$ and $\mathcal{W}_2 = \mathbb{H}^1([0, 2\pi], \mathbb{R}^{n_2})$.

Fact 3.1. *Under the above assumptions $P\nabla\eta(u) = o(\|u_1\|_{\mathbb{H}^1_{2\pi}})$, as $\|u_1\|_{\mathbb{H}^1_{2\pi}} \to \infty$ uniformly under bounded subsets of $\mathcal{W}_2$, i.e.*

$$\forall_{R_2>0} \forall_{\epsilon>0}\ \exists_{R_1>0}\ \forall_{u=(u_1,u_2)\in\mathbb{H}^1_{2\pi}} \quad \|u_2\|_{\mathbb{H}^1_{2\pi}} \le R_2 \ \wedge \|u_1\|_{\mathbb{H}^1_{2\pi}} > R_1 \Rightarrow \ \|P\nabla\eta(u)\|_{\mathbb{H}^1_{2\pi}} < \epsilon \|u_1\|_{\mathbb{H}^1_{2\pi}}. \tag{3.11}$$

Proof. Consider $\langle P\nabla\eta(u), v_1 \rangle = \langle \nabla\eta(u), v_1 \rangle = \int_0^{2\pi} (W'_{x_1}(u(t)), v_1(t))\, dt$. From **A.4** we obtain that

$$\forall_{R>0} \forall_{r>0}\ \exists_{C(R,r)>0}\ \forall_{x\in\mathbb{R}^n} \quad |x_2| \le R \Rightarrow |W'_{x_1}(x)| \le r|x_1| + C(R, r). \tag{3.12}$$

Obviously, there exists constant $C > 0$ such that for every $u \in \mathbb{H}^1_{2\pi}$ the following estimation is true

$$\|u\|_{C^0_{2\pi}} \le C\|u\|_{\mathbb{H}^1_{2\pi}}, \tag{3.13}$$

see [10]. Let $r > 0$, $R > 0$. Choose $C(R, r)$ from the condition (3.12). For every $u \in \mathbb{H}^1_{2\pi}$ such that $\|u_2\|_{\mathbb{H}^1_{2\pi}} \le \frac{1}{C}R$ we have $\|u\|_{C^0_{2\pi}} \le R$ and from (3.12) we obtain

$$|\langle P\nabla\eta(u), v_1 \rangle| \ \le\ \left(\int_0^{2\pi} (r|u_1(t)| + C(R, r))^2\, dt\right)^{\frac{1}{2}} \left(\int_0^{2\pi} |v_1(t)|^2\, dt\right)^{\frac{1}{2}}.$$

Obviously $(\int_0^{2\pi} |v_1(t)|^2\, dt)^{\frac{1}{2}} \leq \|v_1\|_{\mathbb{H}^1_{2\pi}}$, whereas the first integral we estimate in the following way

$$\left(\int_0^{2\pi} (r|u_1(t)| + C(R,r))^2\, dt\right)^{\frac{1}{2}} \leq \left(\int_0^{2\pi} 3r^2|u_1(t)|^2 + 3C(R,r)^2\, dt\right)^{\frac{1}{2}}$$
$$\leq \sqrt{3}\,(r^2\|u_1\|^2_{\mathbb{H}^1_{2\pi}} + 2\pi C(R,r)^2)^{\frac{1}{2}}$$
$$\leq \sqrt{3}\,(r\|u_1\|_{\mathbb{H}^1_{2\pi}} + C_1(R,r)).$$

Finally we get:

$$\forall_{R>0}\, \forall_{r>0}\, \exists_{C_1(R,r)>0}\, \forall_{v_1\in \mathcal{W}_1}\, \forall_{u=(u_1,u_2)\in \mathbb{H}^1_{2\pi}} \quad \|u_2\|_{\mathbb{H}^1_{2\pi}} \leq \frac{1}{C}R$$
$$\Rightarrow |\langle P\nabla\eta(u), v_1\rangle| \leq \sqrt{3}\,(r\|u_1\|_{\mathbb{H}^1_{2\pi}} + C_1(R,r))\|v_1\|_{\mathbb{H}^1_{2\pi}}, \tag{3.14}$$

where $C > 0$ is constant given by (3.13). We will show now that condition (3.11) is satisfied.
Fix arbitrary $R_2, \epsilon > 0$. Set $R = CR_2$ and $r = \frac{\epsilon}{2\sqrt{3}}$. From condition (3.14) there exists $C_1(R,r) = C_1(\epsilon, R_2) > 0$ such that for every $v_1 \in \mathcal{W}_1$ and $u \in \mathbb{H}^1_{2\pi}$ such that $\|u_2\|_{\mathbb{H}^1_{2\pi}} \leq \frac{1}{C}R$ we have

$$|\langle P\nabla\eta(u), v_1\rangle| \leq \left(\frac{\epsilon}{2}\|u_1\|_{\mathbb{H}^1_{2\pi}} + \sqrt{3}\,C_1(\epsilon,R_2)\right)\,\|v_1\|_{\mathbb{H}^1_{2\pi}}.$$

For $v_1 = P\nabla\eta(u)$, we obtain:

$$\|P\nabla\eta(u)\|^2_{\mathbb{H}^1_{2\pi}} \leq \left(\frac{\epsilon}{2}\|u_1\|_{\mathbb{H}^1_{2\pi}} + \sqrt{3}\,C_1(\epsilon,R_2)\right)\,\|P\nabla\eta(u)\|_{\mathbb{H}^1_{2\pi}}.$$

If $P\nabla\eta(u) = 0$, then obviously $0 = \|P\nabla\eta(u)\|_{\mathbb{H}^1_{2\pi}} < \epsilon\|u_1\|_{\mathbb{H}^1_{2\pi}}$. Suppose then $P\nabla\eta(u) \neq 0$. Moreover assume that $\|u_1\|_{\mathbb{H}^1_{2\pi}} \neq 0$. Then

$$\frac{\|P\nabla\eta(u)\|_{\mathbb{H}^1_{2\pi}}}{\|u_1\|_{\mathbb{H}^1_{2\pi}}} \leq \left(\frac{\epsilon}{2} + \frac{\sqrt{3}\,C_1(\epsilon,R_2)}{\|u_1\|_{\mathbb{H}^1_{2\pi}}}\right).$$

Thus there exists $R_1 > 0$, $R_1 = \frac{2\sqrt{3}\,C_1(\epsilon,R_2)}{\epsilon}$, such that if $\|u_1\|_{\mathbb{H}^1_{2\pi}} > R_1$, $(\frac{1}{\|u_1\|_{\mathbb{H}^1_{2\pi}}} < \frac{1}{R_1})$ then

$$\frac{\|P\nabla\eta(u)\|_{\mathbb{H}^1_{2\pi}}}{\|u_1\|_{\mathbb{H}^1_{2\pi}}} \leq \left(\frac{\epsilon}{2} + \sqrt{3}\,C_1(\epsilon,R_2)\,\frac{\epsilon}{2\sqrt{3}\,C_1(\epsilon,R_2)}\right) = \epsilon,$$

which ends the proof. □

Now we are going to define S^1-invariant admissible gradient homotopy $F : \mathbb{H}^1_{2\pi} = (\mathcal{W}_1 \oplus \mathcal{W}_2) \times [0,1] \to \mathbb{R}$ which allows us to compute index of infinity for partially asymptotically linear Newtonian systems (3.1):

$$F(u,s) = \frac{1}{2}\int_0^{2\pi} |\dot{u}(t)|^2\, dt - s\int_0^{2\pi} V(u(t))\, dt - (1-s)\int_0^{2\pi} \frac{1}{2}(Bu_1(t), u_1(t)) - V_2(u_2(t))\, dt. \tag{3.15}$$

This homotopy has been introduced by Jiang in [13].

Lemma 3.1. *We assume that the potential $V \in C^2(\mathbb{R}^n, \mathbb{R})$ satisfies all the assumptions* **A.1–A.5**. *Then there exists $R > 0$ and S^1-invariant gradient homotopy $F : (\mathcal{W}_1 \oplus \mathcal{W}_2) \times [0,1] \to \mathbb{R}$, satisfying the following conditions:*

(1) $(\nabla F)^{-1}(0) \cap ((\mathbb{H}^1_{2\pi} \setminus B_R(\mathbb{H}^1_{2\pi})) \times [0,1]) = \emptyset$,
(2) $\nabla F((u_1,u_2),s) = (Id - \nabla g_s)(u_1,u_2)$ *for* $s \in [0,1]$, $(u_1,u_2) \in \mathcal{W}_1 \oplus \mathcal{W}_2$, *where* $\nabla g_s(\cdot) = \nabla g(\cdot, s)$ *and* $\nabla g : \mathbb{H}^1_{2\pi} \times [0,1] \to \mathbb{H}^1_{2\pi}$ *is compact operator,*
(3) $\nabla F((u_1,u_2),1) = \nabla \Phi_V(u_1,u_2)$,
(4) $\nabla F((u_1,u_2),0) = J_1(u_1) + J_2(u_2)$, *where* $J_i : \mathcal{W}_i \to \mathbb{R}$, $i = 1, 2$ *are defined by formulas* (3.4), (3.5).

Proof. (1) Consider homotopy (3.15) and notice that

$$F(u,s) = F((u_1,u_2),s)$$
$$= \frac{1}{2}\int_0^{2\pi} |\dot{u}(t)|^2\, dt - \int_0^{2\pi} \frac{1}{2}(Bu_1(t), u_1(t)) + V_2(u_2(t))\, dt - s\int_0^{2\pi} W(u(t))\, dt \tag{3.16}$$

and

$$\langle \nabla F((u_1,u_2),s), v\rangle = \int_0^{2\pi} (\dot u(t), \dot v(t)) - (Bu_1(t), v_1(t)) - (V_2'(u_2(t)), v_2(t)) - s(W'(u(t)), v(t))\, dt,$$

where $u, v \in \mathbb{H}^1_{2\pi}$. Obviously

$$\nabla F((u_1,u_2),s) = 0 \Leftrightarrow \begin{cases} P\nabla F((u_1,u_2),s) = 0, \\ (I-P)\nabla F((u_1,u_2),s) = 0. \end{cases}$$

Firstly we consider the second equation $(I-P)\nabla F((u_1,u_2),s) = 0$. Note that from the assumption **A.3** and **A.5** we have:

$$\begin{aligned}\langle (I-P)\nabla F((u_1,u_2),s), u_2\rangle &= \langle \nabla F((u_1,u_2),s), u_2\rangle \\ &= \int_0^{2\pi} |\dot u_2(t)|^2 - (V_2'(u_2(t)), u_2(t)) - s(W'_{x_2}(u(t)), u_2(t))\, dt \\ &\geq \int_0^{2\pi} |\dot u_2(t)|^2 + |u_2(t)|^2\, dt - s\int_0^{2\pi} (W'_{x_2}(u(t)), u_2(t))\, dt \\ &\geq \|u_2\|^2_{\mathbb{H}^1_{2\pi}} - 2\pi M.\end{aligned}$$

Thus if $\|u_2\|_{\mathbb{H}^1_{2\pi}} > R > \sqrt{2\pi M}$, then $(I-P)\nabla F((u_1,u_2),s) \neq 0$.

Consider now the first equation $P\nabla F((u_1,u_2),s) = 0$. Notice that

$$\langle P\nabla F((u_1,u_2),s), v_1\rangle = \langle \nabla J_1(u_1) - sP\nabla\eta(u), v_1\rangle.$$

Fix $R > \sqrt{2\pi M}$. Set $R_2 = R$ and $\epsilon = \frac{\|(Id-L)^{-1}\|^{-1}_{\mathbb{H}^1_{2\pi}}}{2}$. From the condition (3.11) there exists $R_1 > 0$ such that for $u = (u_1,u_2) \in \mathcal{W}_1 \oplus \mathcal{W}_2 = \mathbb{H}^1_{2\pi}$ such that $\|u_1\|_{\mathbb{H}^1_{2\pi}} > R_1$ and $\|u_2\|_{\mathbb{H}^1_{2\pi}} \leq R_2$ we have:

$$\begin{aligned}\|P\nabla F((u_1,u_2),s)\|_{\mathbb{H}^1_{2\pi}} &= \|(Id-L)u_1 - sP\nabla\eta(u)\|_{\mathbb{H}^1_{2\pi}} \\ &\geq \|(Id-L)u_1\|_{\mathbb{H}^1_{2\pi}} - s\|P\nabla\eta(u)\|_{\mathbb{H}^1_{2\pi}} \\ &\geq \|(Id-L)u_1\|_{\mathbb{H}^1_{2\pi}} - \|P\nabla\eta(u)\|_{\mathbb{H}^1_{2\pi}} \\ &\geq \|(Id-L)^{-1}\|^{-1}_{\mathbb{H}^1_{2\pi}} \|u_1\|_{\mathbb{H}^1_{2\pi}} - \varepsilon\|u_1\|_{\mathbb{H}^1_{2\pi}} \\ &\geq \frac{\|(Id-L)^{-1}\|^{-1}_{\mathbb{H}^1_{2\pi}}}{2}\|u_1\|_{\mathbb{H}^1_{2\pi}} > 0.\end{aligned}$$

Thus $(\nabla F)^{-1}(0) \subset B_{R_1}(\mathcal{W}_1) \times B_{R_2}(\mathcal{W}_2) \times [0,1]$.

(2) Notice that $\nabla F(u,s) = \nabla F((u_1,u_2),s) = u - \nabla g((u_1,u_2),s) = u - Lu_1 - \nabla\xi(u_2) - s\nabla\eta(u)$, where operators L, ξ, η are defined respectively by the formulas (3.6)–(3.8). Thus $\nabla g : \mathbb{H}^1_{2\pi} \times [0,1] \to \mathbb{H}^1_{2\pi}$ is compact operator.

For the proof of (3) and (4) note that $F(u,1) = \Phi_V(u)$, and

$$\begin{aligned}F(u,0) &= \frac{1}{2}\int_0^{2\pi} |\dot u(t)|^2\, dt - \int_0^{2\pi} \frac{1}{2}(Bu_1,u_1) - V_2(u_2)\, dt \\ &= \frac{1}{2}\int_0^{2\pi} |\dot u_1(t)|^2 - (Bu_1,u_1)\, dt + \int_0^{2\pi} \frac{1}{2}|\dot u_2(t)|^2 - V_2(u_2)\, dt \\ &= J_1(u_1) + J_2(u_2). \quad \square\end{aligned}$$

For $\alpha \in \mathbb{R}$ denote by $\mu_A(\alpha)$ the multiplicity of α considered as an eigenvalue of matrix A. If $\alpha \notin \sigma(A)$, then it is understood that $\mu_A(\alpha) = 0$.

Definition 3.1. For every $k \in \mathbb{N} \cup \{0\}$ define

(1) $\sigma_k(A, 2\pi) = \sigma(A) \cap (k^2, +\infty)$,
(2) $j_k(A, 2\pi) = \sum_{\alpha\in\sigma_k(A,2\pi)} \mu_A(\alpha)$.

In case $\sigma_k(A, 2\pi) = \emptyset$, we set $j_k(A, 2\pi) = 0$.

We define now the index at infinity $I_V(\infty, 2\pi) \in U(S^1) = \mathbb{Z} \oplus \bigoplus_{k=1}^{\infty} \mathbb{Z}$ for partially asymptotically linear Newtonian systems.

Definition 3.2. We assume that the potential $V \in C^2(\mathbb{R}^n, \mathbb{R})$ is of the form **A.1**. Define $I_V(\infty, 2\pi) \in U(S^1) = \mathbb{Z} \oplus \bigoplus_{k=1}^{\infty} \mathbb{Z}$ by the formula:

$$I_V(\infty, 2\pi)_k = \begin{cases} (-1)^{j_0(B,2\pi)} & \text{for } k = 0, \\ (-1)^{j_0(B,2\pi)} \cdot j_k(B, 2\pi) & \text{for } k \in \mathbb{N}. \end{cases} \tag{3.17}$$

We will show now, that the degree for S^1-equivariant gradient maps of $\nabla\Phi_V$ on $B_R(\mathbb{H}^1_{2\pi})$, where $R > 0$ is sufficiently large radius, is equal to index $I_V(\infty, 2\pi)$.

Lemma 3.2. *We assume that the potential $V \in C^2(\mathbb{R}^n, \mathbb{R})$ satisfies all the assumptions* **A.1–A.5**. *Then $(\nabla\Phi_V)^{-1}(0)$ is bounded and*

$$\nabla_{S^1}\text{-deg}(\nabla\Phi_V, B_R(\mathbb{H}^1_{2\pi})) = I_V(\infty, 2\pi),$$

where functional Φ_V is defined by the formula (3.3) and $R > 0$ is sufficiently large.

Proof. Let $F : (\mathcal{W}_1 \oplus \mathcal{W}_2) \times [0, 1] \to \mathbb{R}$ be S^1-invariant gradient homotopy given by Lemma 3.1. Let $R > 0$ be such that $(\nabla F)^{-1}(0) \cap ((\mathbb{H}^1_{2\pi} \setminus B_R(\mathbb{H}^1_{2\pi})) \times [0, 1]) = \emptyset$. From the homotopy invariance and the product formula of the degree for S^1-equivariant gradient maps, we obtain

$$\begin{aligned} \nabla_{S^1}\text{-deg}(\nabla\Phi_V, B_R(\mathbb{H}^1_{2\pi})) &= \nabla_{S^1}\text{-deg}(\nabla F(\cdot, 1), B_R(\mathbb{H}^1_{2\pi})) \\ &= \nabla_{S^1}\text{-deg}(\nabla F(\cdot, 0), B_R(\mathbb{H}^1_{2\pi})) = \nabla_{S^1}\text{-deg}((\nabla J_1, \nabla J_2), B_R(\mathbb{H}^1_{2\pi})) \\ &= \nabla_{S^1}\text{-deg}(\nabla J_1, B_R(\mathcal{W}_1)) \star \nabla_{S^1}\text{-deg}(\nabla J_2, B_R(\mathcal{W}_2)). \end{aligned} \tag{3.18}$$

Since the operator $\nabla J_1 = Id - L$ is isomorphism we get:

$$\nabla_{S^1}\text{-deg}(\nabla J_1, B_R(\mathcal{W}_1))_H = \begin{cases} (-1)^{j_0(B,2\pi)} & \text{for } H = S^1, \\ (-1)^{j_0(B,2\pi)} \cdot j_k\,(B, 2\pi) & \text{for } H = \mathbb{Z}_k, \end{cases} \tag{3.19}$$

where $\sigma_k(B, 2\pi)$ and $j_k(B, 2\pi)$ are given by the formula (3.1) (see Fact 5.1.1. in [17]).

Note that operator ∇J_2, where $J_2(u_2) = \int_0^{2\pi} \frac{1}{2}|\dot{u}_2(t)|^2 - V_2(u_2)\ dt$, is homotopic to the identity. Indeed, define homotopy $J(u_2, s) = sJ_2(u_2) + \frac{1}{2}(1-s)\|u_2\|^2_{\mathbb{H}^1_{2\pi}}$. Then $J(\cdot, 0) = \frac{1}{2}\|u_2\|^2_{\mathbb{H}^1_{2\pi}}$,$J(\cdot, 1) = J_2$ and

$$\begin{aligned} \langle \nabla J(u_2, s), u_2\rangle &= \langle s\nabla J_2(u_2) + (1-s)u_2, u_2\rangle \\ &\geq s\int_0^{2\pi} |\dot{u_2}|^2 + |u_2|^2\ dt + (1-s)\|u_2\|^2_{\mathbb{H}^1_{2\pi}} = \|u_2\|^2_{\mathbb{H}^1_{2\pi}}. \end{aligned}$$

Hence

$$\nabla_{S^1}\text{-deg}(\nabla J_2, B_R(\mathcal{W}_2)) = \nabla_{S^1}\text{-deg}(Id, B_R(\mathcal{W}_2)) = \mathbb{I} \in U(S^1). \tag{3.20}$$

Summing up, from (3.18)–(3.20) we obtain that

$$\nabla_{S^1}\text{-deg}(\nabla\Phi_V, B_R(\mathbb{H}^1_{2\pi})) = \nabla_{S^1}\text{-deg}(Id - L, B_R(\mathcal{W}_1)) = I_V(\infty, 2\pi),$$

which completes the proof. □

4. Existence and connected sets of periodic solutions bifurcating from infinity

In this section we formulate sufficient conditions for the existence, continuation and bifurcation from infinity of periodic solutions of the partially asymptotically linear Newtonian systems.

We make additional assumption:

A.6 $(V')^{-1}(0) = \{p_1, \dots, p_q\}$.

Let $p \in \{p_1, \dots, p_q\}$ and $V \in C^2(\mathbb{R}^n, \mathbb{R})$ satisfies all the assumptions **A.1–A.6**. Note that since $(V')^{-1}(0) < \infty$, the Brouwer degree $\deg_B(-V', B_\alpha(\mathbb{R}^n, p_i), 0)$ for $i = 1, \dots, q$ is well-defined for sufficiently small $\alpha > 0$. Set $\text{ind}(-V', p_i) = \lim_{\alpha\to 0}\deg_B(-V', B_\alpha(\mathbb{R}^n, p_i), 0)$ for $i = 1, \dots, q$. In such situation we have defined in [17] the index $I_V(p, 2\pi) = (I_V(p, 2\pi)_{S^1}, I_V(p, 2\pi)_{\mathbb{Z}_1}, \dots, I_V(p, 2\pi)_{\mathbb{Z}_k}, \dots,) \in U(S^1)$ by the formula

$$I_V(p, 2\pi)_k = \begin{cases} \text{ind}(-V', p) & \text{for } k = 0, \\ \text{ind}(-V', p) \cdot j_k(V''(p), 2\pi) & \text{for } k \in \mathbb{N}. \end{cases} \tag{4.1}$$

In the following two theorems we have obtained the existence of at least one nonstationary periodic solution of partially asymptotically linear Newtonian system. To prove these results we combine Lemma 3.2 of the previous section with Lemmas 5.2.2., 5.2.3. in [17].

Theorem 4.1. *Suppose that all the assumptions* **A.1–A.6** *are satisfied and that $\sigma(V''(p)) \cap \{k^2 : k \in \mathbb{N}\} = \emptyset$ for $p \in \{p_1, \dots, p_q\}$. Additionally suppose that there exists $k \in \mathbb{N}$ such that*

$$(-1)^{j_0(B,2\pi)} \cdot j_k(B, 2\pi) \neq \sum_{p \in \{p_1,\dots,p_q\}} \operatorname{ind}(-V', p) \cdot j_k(V''(p), 2\pi).$$

Then there exists at least one nonstationary 2π-periodic solution of (3.1).

Proof. Consider the functional Φ_V defined by the formula (3.3), which critical points are in one-to-one correspondence with solutions of the system (3.1). According to Lemma 3.1 the set $\nabla\Phi_V^{-1}(0)$ is bounded, thus ∞ is an isolated critical point of the functional Φ_V. Moreover from Lemma 3.2 we get

$$I_V(\infty, 2\pi)_k = (-1)^{j_0(B,2\pi)} \cdot j_k(B, 2\pi) = \nabla_{S^1}\text{-deg}_{\mathbb{Z}_k}(\nabla\Phi_V, B_R(\mathbb{H}^1_{2\pi})).$$

On the other hand from the assumption $\sigma(V''(p)) \cap \{k^2 : k \in \mathbb{N}\} = \emptyset$ for $p \in \{p_1, \dots, p_q\}$ and Lemma 5.2.1. in [17] follows that $p_1, \dots, p_q$ are isolated critical points of Φ_V. From Lemma 5.2.2. in [17] the degree $\nabla_{S^1}\text{-deg}(\nabla\Phi_V, B_{\alpha_p}(\mathbb{H}^1_{2\pi}, p))$ is defined for sufficiently small radius $\alpha_p > 0$ and it is equal to the index $I_V(p, 2\pi)$ given by (4.1). Hence $\nabla_{S^1}\text{-deg}_{\mathbb{Z}_k}(\nabla\Phi_V, B_{\alpha_p}(\mathbb{H}^1_{2\pi}, p)) = \operatorname{ind}(-V', p_i) \cdot j_k(V''(p_i), 2\pi)$. Arguing by contradiction, we suppose that $p_1, \dots, p_q$ are the only 2π-periodic solutions of (3.1). Thus we can choose $\alpha_\infty, \alpha_{p_i} > 0$, $i = 1, \dots, q$ such that

(i) $(\nabla\Phi_V)^{-1}(0) \cap (\mathbb{H}^1_{2\pi} \setminus B_{\alpha_\infty}(\mathbb{H}^1_{2\pi})) = \emptyset$,
(ii) $(\nabla\Phi_V)^{-1}(0) \cap B_{\alpha_{p_i}}(\mathbb{H}^1_{2\pi}, p_i) = \{p_i\}$, $i = 1, \dots, q$,
(iii) $B_{\alpha_{p_i}}(\mathbb{H}^1_{2\pi}, p_i) \cap B_{\alpha_{p_j}}(\mathbb{H}^1_{2\pi}, p_j) = \emptyset$, for $i \neq j$,
(iv) $cl(B_{\alpha_{p_i}}(\mathbb{H}^1_{2\pi}, p_i)) \subset B_{\alpha_\infty}(\mathbb{H}^1_{2\pi})$ for $i = 1, \dots, q$.

From properties of the degree for S^1-equivariant gradient maps we obtain

$$\nabla_{S^1}\text{-deg}(\nabla\Phi_V, B_{\alpha_\infty}(\mathbb{H}^1_{2\pi})) = \sum_{i=1}^{q} \nabla_{S^1}\text{-deg}(\nabla\Phi_V, B_{\alpha_{p_i}}(\mathbb{H}^1_{2\pi}, p_i)).$$

However, from the assumption we obtain that

$$\begin{aligned}\nabla_{S^1}\text{-deg}_{\mathbb{Z}_k}(\nabla\Phi_V, B_{\alpha_\infty}(\mathbb{H}^1_{2\pi})) &= I_V(\infty, T)_k = (-1)^{j_0(B,2\pi)} \cdot j_k(B, 2\pi) \\ &\neq \sum_{p \in \{p_1,\dots,p_q\}} \operatorname{ind}(-V', p) \cdot j_k(V''(p), 2\pi) \\ &= \sum_{i=1}^{q} \nabla_{S^1}\text{-deg}_{\mathbb{Z}_k}(\nabla\Phi_V, B_{\alpha_{p_i}}(\mathbb{H}^1_{2\pi}, p_i)).\end{aligned}$$

Thus $\nabla_{S^1}\text{-deg}(\nabla\Phi_V, B_{\alpha_\infty}(\mathbb{H}^1_{2\pi})) \neq \sum_{i=1}^{q} \nabla_{S^1}\text{-deg}(\nabla\Phi_V, B_{\alpha_{p_i}}(\mathbb{H}^1_{2\pi}, p_i))$, a contradiction. $\square$

Definition 4.1. Let $A, A_1, \dots, A_q$ be real $(n \times n)$-symmetric matrix.

(i) Let $\sigma(A) \cap \{k^2 : k \in \mathbb{N} \cup \{0\}\} = \{k_1^2, \dots, k_r^2\}$. Put

$$\mathbb{K}(A) = \bigcup_{\{i_1,\dots,i_s\} \in \{1,\dots,r\}} \{gcd(k_{i_1}, \dots, k_{i_s})\}.$$

In the case $\sigma(A) \cap \{k^2 : k \in \mathbb{N} \cup \{0\}\} = \emptyset$, we set $\mathbb{K}(A) = \emptyset$.

(ii) Let $(\sigma(A_1) \cup \dots \cup \sigma(A_q)) \cap \{k^2 : k \in \mathbb{N} \cup \{0\}\} = \{k_1^2, \dots, k_r^2\}$. Put

$$\mathbb{K}(A_1, \dots, A_q) = \bigcup_{\{i_1,\dots,i_s\} \in \{1,\dots,r\}} \{gcd(k_{i_1}, \dots, k_{i_s})\}.$$

In the case $(\sigma(A_1) \cup \dots \cup \sigma(A_q)) \cap \{k^2 : k \in \mathbb{N} \cup \{0\}\} = \emptyset$, we set $\mathbb{K}(A_1, \dots, A_q) = \emptyset$.

Theorem 4.2. *Suppose that all the assumptions* **A.1–A.6** *are satisfied. Suppose that there exists $k \in \mathbb{N} \setminus \mathbb{K}(V''(p_1), \dots, V''(p_q))$ such that*

$$(-1)^{j_0(B,2\pi)} \cdot j_k(B, 2\pi) \neq \sum_{p \in \{p_1,\dots,p_q\}} \operatorname{ind}(-V', p_i) \cdot j_k\left(V''(p_i), 2\pi\right).$$

Then there exists at least one 2π-periodic solutions of the system (3.1). Moreover if $p_0 \in \{p_1, \dots, p_q\}$ is not isolated 2π-periodic solution of the system (3.1), then there exist numbers $k_1, \dots, k_r \in \mathbb{N}$ such that $\sigma(V''(p_0)) \cap \{k^2 : k \in \mathbb{N}\} = \{k_1^2, \dots, k_r^2\}$ and minimal period of arbitrary solution sufficiently close to p_0 equals $\frac{2\pi}{gcd(k_{i_1},\dots,k_{i_s})}$, for some $\{i_1, \dots, i_s\} \subset \{1, \dots, r\}$.

Proof. Note that as in the proof of the previous theorem, infinity is an isolated critical point of functional Φ_V and from Lemma 3.2 we obtain that

$$I_V(\infty, 2\pi)_k = (-1)^{j_0(B,2\pi)} \cdot j_k(B, 2\pi) = \nabla_{S^1}\text{-deg}_{\mathbb{Z}_k}(\nabla\Phi_V, B_R(\mathbb{H}^1_{2\pi})).$$

Moreover, if stationary solutions $p_1, \dots, p_q$ are isolated critical points of the functional Φ_V given by the formula Eq. (3.3), then all the assumption of Lemma 5.2.3. in [17] are satisfied. Thus for $k \in \mathbb{N} \setminus \mathbb{K}(V''(p_1), \dots, V''(p_q))$ we obtain

$$\nabla_{S^1}\text{-deg}_{\mathbb{Z}_k}(\nabla\Phi_V, B_{\alpha_{p_i}}(\mathbb{H}^1_{2\pi}, p_i)) = I_V(p_i, 2\pi)_k = \text{ind}(-V', p_i) \cdot j_k(V''(p_i), 2\pi).$$

Assume that stationary solutions $p_1, \dots, p_q$ are isolated critical points of Φ_V. If they were not isolated, we would obtain an infinite sequence of critical points of Φ_V, i.e. a sequence of 2π-periodic nonstationary solutions of (3.1), which is our assertion. Arguing by contradiction, we suppose that $p_1, \dots, p_q$ are the only 2π-periodic solutions of (3.1). As in the proof of Theorem 4.1 we obtain

$$\nabla_{S^1}\text{-deg}(\nabla\Phi_V, B_{\alpha_\infty}(\mathbb{H}^1_{2\pi})) = \sum_{i=1}^{q} \nabla_{S^1}\text{-deg}(\nabla\Phi_V, B_{\alpha_{p_i}}(\mathbb{H}^1_{2\pi}, p_i)).$$

But on the other hand we have

$$\begin{aligned}\nabla_{S^1}\text{-deg}_{\mathbb{Z}_k}(\nabla\Phi_V, B_{\alpha_\infty}(\mathbb{H}^1_{2\pi})) &= I_V(\infty, 2\pi)_{\mathbb{Z}_k} \\ &\neq \sum_{i=1}^{q} I_V(p_i, 2\pi)_{\mathbb{Z}_k} = \sum_{i=1}^{q} \nabla_{S^1}\text{-deg}_{\mathbb{Z}_k}(\nabla\Phi_V, B_{\alpha_{p_i}}(\mathbb{H}^1_{2\pi}, p_i)).\end{aligned}$$

Hence, $\nabla_{S^1}\text{-deg}(\nabla\Phi, B_{\alpha_\infty}(\mathbb{H}^1_{2\pi})) \neq \sum_{i=1}^{q} \nabla_{S^1}\text{-deg}(\nabla\Phi_V, B_{\alpha_{p_i}}(\mathbb{H}^1_{2\pi}, p_i))$ is a contradiction. The rest of the proof is the same as in the proof of Theorem 5.2.2 in [17]. □

We apply the above existence result to obtain continuation of nonstationary 2π-periodic solutions of the family of autonomous Newtonian systems of the form:

$$(F)_\lambda \begin{cases} \ddot{u}(t) = -V'(u(t), \lambda) \\ u(0) = u(2\pi) \\ \dot{u}(0) = \dot{u}(2\pi) \end{cases} \tag{4.2}$$

where $V \in C^2(\mathbb{R}^n \times \mathbb{R}, \mathbb{R})$. Consider functional $\Phi_V : \mathbb{H}^1_{2\pi} \times \mathbb{R} \to \mathbb{R}$ given by the formula

$$\Phi_V(u, \lambda) = \frac{1}{2}\int_0^{2\pi} |\dot{u}(t)|^2\, dt - \int_0^{2\pi} V(u(t), \lambda)\, dt, \tag{4.3}$$

which critical points are in one-to-one correspondence with solutions of the family (4.2).

Theorem 4.3. *Suppose that $V_0 := V(\cdot, 0)$ satisfies assumptions* **A.1–A.6** *and there exists $k \in \mathbb{N} \setminus \mathbb{K}(V_0''(p_1), \dots, V_0''(p_q))$ such that*

$$(-1)^{j_0(B,2\pi)} \cdot j_k(B, 2\pi) \neq \sum_{p \in \{p_1, \dots, p_q\}} \text{ind}(-V_0', p) \cdot j_k(V_0''(p), 2\pi). \tag{4.4}$$

Then there exists an infinite sequence of 2π-periodic solutions of the system $(F)_0$ convergent to some $p \in \{p_1, \dots, p_q\}$ or there exist closed connected sets $\mathcal{C}^\pm$ such that

$$\mathcal{C}^- \subset (\mathbb{H}^1_{2\pi} \times (-\infty, 0]) \cap (\nabla\Phi_V)^{-1}(0),$$
$$\mathcal{C}^+ \subset (\mathbb{H}^1_{2\pi} \times [0, +\infty)) \cap (\nabla\Phi_V)^{-1}(0).$$

Moreover for $\mathcal{C} = \mathcal{C}^\pm$

(C1) $\mathcal{C} \cap ((B_{\alpha_\infty}(\mathbb{H}^1_{2\pi}) \setminus \bigcup_{i=1}^{q} D_{\alpha_{p_i}}(\mathbb{H}^1_{2\pi}, p_i)) \times \{0\}) \neq \emptyset$,
(C2) *or $\mathcal{C}$ is unbounded or $\mathcal{C} \cap \{p_1, \dots, p_q\} \neq \emptyset$.*

Additionally if $p_0 \in \{p_1, \dots, p_q\}$ is not isolated 2π-periodic solution of the system $(F)_0$, then there exist numbers $k_1, \dots, k_r \in \mathbb{N}$ such that $\sigma(V_0''(p_0)) \cap \{k^2 : k \in \mathbb{N}\} := \{k_1^2, \dots, k_r^2\}$ and minimal period of arbitrary solution sufficiently close to p_0 equals $\frac{2\pi}{gcd(k_{i_1}, \dots, k_{i_s})}$, for some $\{i_1, \dots, i_s\} \subset \{1, \dots, r\}$.

Proof. Consider functional $\Phi_{V_0} \in C^2_{S^1}(\mathbb{H}^1_{2\pi}, \mathbb{R})$ given by the formula (3.3) and suppose that stationary solutions $p_1, \dots, p_q$ are isolated solutions of (4.2) at level $\lambda_0 = 0$, i.e. are isolated critical points of Φ_{V_0}. Moreover according to Lemma 3.1 the set $\nabla\Phi_{V_0}^{-1}(0)$ is bounded, thus ∞ is an isolated critical point of functional Φ_{V_0}. Therefore we can choose $\alpha_\infty, \alpha_{p_i} > 0$, $i = 1, \dots, q$ such that

(i) $(\nabla\Phi_{V_0})^{-1}(0) \cap (\mathbb{H}^1_{2\pi} \setminus B_{\alpha_\infty}(\mathbb{H}^1_{2\pi})) = \emptyset$,
(ii) $(\nabla\Phi_{V_0})^{-1}(0) \cap B_{\alpha_{p_i}}(\mathbb{H}^1_{2\pi}, p_i) = \{p_i\}$, $i = 1, \ldots, q$,
(iii) $B_{\alpha_{p_i}}(\mathbb{H}^1_{2\pi}, p_i) \cap B_{\alpha_{p_j}}(\mathbb{H}^1_{2\pi}, p_j) = \emptyset$, for $i \neq j$.

Put $\Omega = B_{\alpha_\infty}(\mathbb{H}^1_{2\pi}) \setminus \bigcup_{i=1}^q D_{\alpha_{p_i}}(\mathbb{H}^1_{2\pi}, p_i)$. Notice that from the assumption (4.4), Lemma 3.2 of the previous section and Lemma 5.2.3. in [17] we obtain that $\nabla_{S^1}\text{-deg}(\nabla\Phi_{V_0}, \Omega) \neq \Theta \in U(S^1)$. The rest of the proof is direct consequence of Theorem 2.1. Second part of Theorem one can prove in the same way as in the proof of Theorem 4.2. □

Now we will formulate some corollaries and remarks.

Corollary 4.1. *Let all the assumptions of Theorem 4.3 be satisfied. Additionally suppose that $\sigma(V_0''(p)) \cap \{k^2 : k \in \mathbb{N}\} = \emptyset$ for every $p \in \{p_1, \ldots, p_q\}$. Then there exist closed connected sets $\mathcal{C}^- \subset (\mathbb{H}^1_{2\pi} \times (-\infty, 0]) \cap (\nabla\Phi_V)^{-1}(0)$, $\mathcal{C}^+ \subset (\mathbb{H}^1_{2\pi} \times [0, +\infty)) \cap (\nabla\Phi_V)^{-1}(0)$ with properties* (C1), (C2).

Remark 4.1. If $\mathcal{C} = \mathcal{C}^\pm$ in Theorem 4.3 is bounded, then there appears phenomenon of symmetry breaking, i.e. $\mathcal{C}$ contains solutions with different minimal periods. Indeed, in this case from (C2) we obtain that $\mathcal{C}$ contain stationary solution (which isotropy group in $\mathbb{H}^1_{2\pi}$ is S^1) and from (C1) — nonstationary solution (with isotropy group $\mathbb{Z}_k$ for some $k \in \mathbb{N}$, which means that its minimal period equals $\frac{2\pi}{k}$).

Remark 4.2. Note that we cannot proof Theorem 4.3 using the Leray–Schauder degree. In the proof it was crucial to obtain nontriviality of the degree $\nabla_{S^1}\text{-deg}(\nabla\Phi_{V_0}, \Omega) \neq \Theta \in U(S^1)$, where $\Omega = B_{\alpha_\infty}(\mathbb{H}^1_{2\pi}) \setminus \bigcup_{i=1}^q D_{\alpha_{p_i}}(\mathbb{H}^1_{2\pi}, p_i)$. However applying Theorem 2.3, one can show that the Leray–Schauder degree vanishes:

$$\deg_{\mathrm{LS}}\left(\nabla\Phi_V, B_{\alpha_\infty}(\mathbb{H}^1_{2\pi}) \setminus \bigcup_{i=1}^p D_{\alpha_{p_i}}(\mathbb{H}^1_{2\pi}, p_i), 0\right) = \deg_{\mathrm{B}}\left(-V_0', B_{\alpha_\infty}(\mathbb{R}^n) \setminus \bigcup_{i=1}^p D_{\alpha_{p_i}}\big(\mathbb{R}^n, p_i\big), 0\right) = 0 \in \mathbb{Z},$$

since $(V_0')^{-1}(0) \cap (B_{\alpha_\infty}(\mathbb{R}^n) \setminus \bigcup_{i=1}^p D_{\alpha_{p_i}}(\mathbb{R}^n, p_i)) = \emptyset$.

In the following theorem we prove the sufficient conditions for the existence of connected sets of periodic solutions of the family of partially asymptotically linear Newtonian systems (4.2) bifurcating from infinity.

Theorem 4.4. *Let $V \in C^2(\mathbb{R}^n \times \mathbb{R}, \mathbb{R})$. Suppose that there exist real numbers $\lambda_- < \lambda_+$ such that potentials $V(x, \lambda_\pm) = V(x_1, x_2, \lambda_\pm) = \frac{1}{2}(B(\lambda_\pm)x_1, x_1) + V_2(x_2, \lambda_\pm) + W(x, \lambda_\pm)$, where $x = (x_1, x_2) \in \mathbb{R}^{n_1} \oplus \mathbb{R}^{n_2} = \mathbb{R}^n$ and $B(\lambda_\pm)$ is real $(n \times n)$-symmetric matrix, satisfies assumptions* **A.1–A.5**. *Additionally suppose that at least one of the following conditions is satisfied:*

(i) $(-1)^{j_0(B(\lambda_+),2\pi)} \neq (-1)^{j_0(B(\lambda_-),2\pi)}$,
(ii) *there exists $k \in \mathbb{N}$ such that*

$$(-1)^{j_0(B(\lambda_+),2\pi)} \cdot j_k(B(\lambda_+), 2\pi) \neq (-1)^{j_0(B(\lambda_-),2\pi)} \cdot j_k(B(\lambda_-), 2\pi).$$

Then there exists an unbounded closed connected component $C \subset \mathbb{H}^1_{2\pi} \times [\lambda_-, \lambda_+]$ of the set of solutions of the system (4.2) such that $C \cap (B_\gamma(\mathbb{H}^1_{2\pi}) \times \{\lambda_-, \lambda_+\}) \neq \emptyset$.

Proof. It is clear that the functional given by the formula (4.3) satisfies condition **(c1)**.

(i) From Lemma 3.2, for sufficiently large $\gamma > 0$ we obtain that

$$\nabla_{S^1}\text{-deg}_{S^1}(\nabla_u\Phi_V(\cdot, \lambda_\pm), B_\gamma(\mathbb{H}^1_{2\pi})) = (-1)^{j_0(B(\lambda_\pm),2\pi)}.$$

Therefore

$$\begin{aligned}\mathrm{BIF}_{S^1}(\infty, [\lambda_-, \lambda_+]) &= \nabla_{S^1}\text{-deg}_{S^1}(\nabla_u\Phi_V(\cdot, \lambda_+), B_\gamma(\mathbb{H}^1_{2\pi})) - \nabla_{S^1}\text{-deg}_{S^1}(\nabla_u\Phi_V(\cdot, \lambda_-), B_\gamma(\mathbb{H}^1_{2\pi}))\\ &= (-1)^{j_0(B(\lambda_+),2\pi)} - (-1)^{j_0(B(\lambda_-),2\pi)} \neq 0.\end{aligned}$$

(ii) From Lemma 3.2, for sufficiently large $\gamma > 0$ we obtain that

$$\nabla_{S^1}\text{-deg}_{\mathbb{Z}_k}(\nabla_u\Phi_V(\cdot, \lambda_\pm), B_\gamma(\mathbb{H}^1_{2\pi})) = (-1)^{j_0(B(\lambda_\pm),2\pi)} \cdot j_k(B(\lambda_\pm), 2\pi).$$

Therefore

$$\begin{aligned}\mathrm{BIF}_{\mathbb{Z}_k}(\infty, [\lambda_-, \lambda_+]) &= \nabla_{S^1}\text{-deg}_{\mathbb{Z}_k}(\nabla_u\Phi_V(\cdot, \lambda_+), B_\gamma(\mathbb{H}^1_{2\pi})) - \nabla_{S^1}\text{-deg}_{\mathbb{Z}_k}(\nabla_u\Phi_V(\cdot, \lambda_-), B_\gamma(\mathbb{H}^1_{2\pi}))\\ &= (-1)^{j_0(B(\lambda_+),2\pi)} \cdot j_k(B(\lambda_+), 2\pi) - (-1)^{j_0(B(\lambda_-),2\pi)} \cdot j_k(B(\lambda_-), 2\pi) \neq 0.\end{aligned}$$

Since $\mathrm{BIF}(\infty, [\lambda_-, \lambda_+]) \neq \Theta \in U(S^1)$, the rest of the proof is direct consequence of Theorem 2.2. □

Remark 4.3. Notice that in the paper [18] we have considered the following family of Newtonian systems

$$\begin{cases} \ddot{u}(t) = -V'(u(t), \lambda) \\ u(0) = u(2\pi) \\ \dot{u}(0) = \dot{u}(2\pi) \end{cases} \tag{4.5}$$

for a family of potentials $V \in C^2(\mathbb{R}^n \times \mathbb{R}, \mathbb{R})$ of the form

$$V(x, \lambda) = \frac{1}{2}(A(\lambda)x, x) + \eta(x, \lambda),$$

where $A(\lambda)$ is a real symmetric matrix for every $\lambda \in \mathbb{R}$ and $\nabla_x \eta(x, \lambda) = o(\|x\|)$, as $\|x\| \to \infty$ uniformly on bounded λ-intervals. It is obvious that (4.5) is a family of asymptotically linear systems. In Theorem 4.1. of [18] we have proved the existence of connected sets of periodic solutions of (4.5) bifurcating from infinity. We have admitted in this theorem the resonant case i.e. $\sigma(A(\lambda_\pm)) \cap \{k^2 : k \in \mathbb{N} \cup \{0\}\} \neq \emptyset$. In this paper we study partially asymptotically Newtonian systems (4.2). Therefore, Theorem 4.4 is generalization of Theorem 4.1. of [18] but only in non-resonant case, i.e. when $\sigma(A(\lambda_\pm)) \cap \{k^2 : k \in \mathbb{N} \cup \{0\}\} = \emptyset$.

Now we put stronger assumptions on the family of potentials $V \in C^2(\mathbb{R}^n \times \mathbb{R}, \mathbb{R})$, in order to obtain some additional properties of component C obtained in the Theorem 4.4. Suppose that

B.1 $V \in C^2(\mathbb{R}^n \times \mathbb{R}, \mathbb{R})$ is of the form

$$V(x, \lambda) = V(x_1, x_2, \lambda) = \frac{1}{2}(B(\lambda)x_1, x_1) + V_2(x_2, \lambda) + W(x, \lambda),$$

where $x = (x_1, x_2) \in \mathbb{R}^{n_1} \oplus \mathbb{R}^{n_2} = \mathbb{R}^n$ and $B(\lambda)$ is real $(n_1 \times n_1)$-symmetric matrix for every $\lambda \in \mathbb{R}$.

Moreover assume that for some $\lambda_0 \in \mathbb{R}$ there exist $\lambda_- < \lambda_+$ such that

B.2 $\{\lambda \in [\lambda_-, \lambda_+] : \sigma(B(\lambda)) \cap \{k^2 : k \in \mathbb{N} \cup \{0\}\} \neq \emptyset\} = \{\lambda_0\}$,
B.3 $(-V_2'(x_2, \lambda), x_2) \geq |x_2|^2$ for every $x_2 \in \mathbb{R}^{n_2}$, $\lambda \in [\lambda_-, \lambda_+]$,
B.4 $W'_{x_1}(x, \lambda) = o(|x_1|)$ as $|x_1| \to \infty$ uniformly under compact subsets of $\mathbb{R}^{n_2}$ and uniformly under compact λ-intervals, i.e.

$$\forall_{R_2>0} \forall_{\epsilon>0}\ \exists_{R_1>0}\ \forall_{\lambda \in [a,b] \subset \mathbb{R}}\ \forall_{x \in \mathbb{R}^n} \quad |x_2| < R_2\ \wedge |x_1| > R_1 \Rightarrow\ |W'_{x_1}(x, \lambda)| < \epsilon |x_1|, \tag{4.6}$$

B.5 $\exists_{M>0}\ \forall_{\lambda \in [\lambda_-, \lambda_+]}\ \forall_{x \in \mathbb{R}^n}\ (W'_{x_2}(x, \lambda), x_2) < M$.

Consider system (4.2) with family of potentials $V \in C^2(\mathbb{R}^n \times \mathbb{R}, \mathbb{R})$ satisfying assumptions **B.1**–**B.5** and functional (4.3) corresponding to the system (4.2).

Let us define the following families of mappings

$$J_1 : \mathcal{W}_1 \times \mathbb{R} \to \mathbb{R}, \quad J_1(u_1, \lambda) = \frac{1}{2}\int_0^{2\pi} |\dot{u}_1(t)|^2 - (B(\lambda)u_1(t), u_1(t))\, dt, \tag{4.7}$$

$$J_2 : \mathcal{W}_2 \times \mathbb{R} \to \mathbb{R}, \quad J_2(u_2, \lambda) = \int_0^{2\pi} \frac{1}{2}|\dot{u}_2(t)|^2 - V_2(u_2(t), \lambda)\, dt, \tag{4.8}$$

$$\eta : \mathbb{H}^1_{2\pi} \times \mathbb{R} \to \mathbb{R}, \quad \eta(u, \lambda) = \int_0^{2\pi} W(u(t), \lambda)\, dt. \tag{4.9}$$

Note that

- $\nabla J_1(u_1, \lambda) = (Id - L(\lambda))u_1$, where family of operators $L(\lambda) : \mathcal{W}_1 \to \mathcal{W}_1$ is given by the formula

$$\langle L(\lambda)u_1, v_1\rangle = \int_0^{2\pi} (u_1(t) + B(\lambda)u_1(t), v_1(t))\, dt, \tag{4.10}$$

- $\nabla J_2(u_2, \lambda) = (Id - \nabla\xi_\lambda)u_2$, where family $\xi_\lambda(u_2) = \xi(u_2, \lambda)$ i $\nabla\xi : \mathcal{W}_2 \times \mathbb{R} \to \mathcal{W}_2$ is given by the formula

$$\langle \nabla\xi(u_2, \lambda), v_2\rangle = \int_0^{2\pi} (u_2(t) - V_2'(u_2(t), \lambda), v_2(t))\, dt, \tag{4.11}$$

- $\nabla\eta : \mathbb{H}^1_{2\pi} \times \mathbb{R} \to \mathbb{H}^1_{2\pi}$ is given by the formula

$$\langle \nabla\eta(u, \lambda), v\rangle = \int_0^{2\pi} (W'(u(t), \lambda), v(t))\, dt. \tag{4.12}$$

It is clear that operators defined by the formulas (4.10)–(4.12) are compact. Moreover from assumption **B.2** and Corollary 5.1.1. in [17] linear operator $\nabla J_1(\cdot,\lambda) = Id - L(\lambda)$ is isomorphism for every $\lambda \in [\lambda_-,\lambda_+] \setminus \{\lambda_0\}$.

Now we present the counterpart of Fact 3.1.

Fact 4.1. *Operator $P\nabla\eta(u,\lambda) = o(\|u_1\|_{\mathbb{H}^1_{2\pi}})$, as $\|u_1\|_{\mathbb{H}^1_{2\pi}} \to \infty$ uniformly under bounded subsets of $\mathcal{W}_2$ and uniformly under compact λ-intervals, i.e.*

$$\forall_{R_2>0}\forall_{\epsilon>0}\ \exists_{R_1>0}\ \forall_{\lambda\in[a,b]}\ \forall_{u\in\mathbb{H}^1_{2\pi}}\quad \|u_2\|_{\mathbb{H}^1_{2\pi}} \le R_2\ \wedge \|u_1\|_{\mathbb{H}^1_{2\pi}} > R_1 \Rightarrow\ \|P\nabla F(u)\|_{\mathbb{H}^1_{2\pi}} < \epsilon\|u_1\|_{\mathbb{H}^1_{2\pi}}. \tag{4.13}$$

Proof. Fix interval $[a,b] \subset \mathbb{R}$. Notice that the assumption **B.4** implies that

$$\forall_{R>0}\forall_{r>0}\ \exists_{C>0}\ \forall_{\lambda\in[a,b]}\ \forall_{x\in\mathbb{R}^n}\quad |x_2| \le R \Rightarrow |W'_{x_1}(x,\lambda)| \le r|x_1| + C, \tag{4.14}$$

where constant C depends on $R > 0, r > 0$ and an interval $[a,b]$. Since C does not depend on $\lambda \in [a,b]$, the proof is analogous of the proof of Fact 3.1. $\square$

Definition 4.2. We say that unbounded connected closed set C meets (∞,λ_0), if for every $\delta, \varrho > 0$

$$C \cap \{(\mathbb{H}^1_{2\pi} \setminus B_\varrho(\mathbb{H}^1_{2\pi})) \times [\lambda_0-\delta,\lambda_0+\delta]\} \ne \emptyset. \tag{4.15}$$

Theorem 4.5. *Let $V \in C^2(\mathbb{R}^n \times \mathbb{R},\mathbb{R})$ satisfies assumptions* **B.1–B.5**. *Additionally suppose that at least one of the following condition is satisfied:*

(i) $(-1)^{j_0(B(\lambda_+),2\pi)} \ne (-1)^{j_0(B(\lambda_-),2\pi)}$,

(ii) *there exists $k \in \mathbb{N}$ such that*

$$(-1)^{j_0(B(\lambda_+),2\pi)} \cdot j_k(B(\lambda_+),2\pi) \ne (-1)^{j_0(B(\lambda_-),2\pi)} \cdot j_k(B(\lambda_-),2\pi).$$

Then there exists an unbounded closed connected component $C \subset \mathbb{H}^1_{2\pi} \times [\lambda_-,\lambda_+]$ of the set of solutions of system (4.2) such that $C \cap (B_\gamma(\mathbb{H}^1_{2\pi}) \times \{\lambda_-,\lambda_+\}) \ne \emptyset$. Moreover component C meets (∞,λ_0).

Proof. Notice that potentials $V(\cdot,\lambda_\pm)$ satisfies assumptions **A.1–A.5**. Thus the existence of an unbounded closed connected component $C \subset \mathbb{H}^1_{2\pi} \times [\lambda_-,\lambda_+]$ we obtain as a direct consequence of Theorem 4.4. To the end of the proof it remains to show that component C meets (∞,λ_0). We will show that $C \cap ((\mathbb{H} \setminus B_\varrho(\mathbb{H},\infty)) \times [\lambda_-,\lambda_0-\delta) \cup (\lambda_0+\delta,\lambda_+]) = \emptyset$ for large $\varrho > 0$ and small $\delta > 0$. In consequence, since component C is unbounded, we obtain that $C \cap (\mathbb{H} \setminus B_\varrho(\mathbb{H},\infty)) \times [\lambda_0-\delta,\lambda_0+\delta] \ne \emptyset$, which is enough to show the condition in the Definition 4.2.

Fix $\delta > 0$ and denote by $A := [\lambda_-,\lambda_0-\delta] \cup [\lambda_0+\delta,\lambda_+]$. We will show that $\nabla\Phi_V(u,\lambda) \ne 0$ for $\lambda \in A$ and $\|u\|_{\mathbb{H}^1_{2\pi}} > \varrho$, where $\varrho > 0$ is sufficiently large. Note that analogously as in the proof of Lemma 3.1, from assumption **B.3** and **B.5** we obtain that $(I-P)\nabla\Phi_V(u,\lambda) \ne 0$ for $\|u_2\|_{\mathbb{H}^1_{2\pi}} > R = 2\pi M$. Since $P\nabla\eta(u,\lambda) = o(\|u_1\|_{\mathbb{H}^1_{2\pi}})$, as $\|u_1\|_{\mathbb{H}^1_{2\pi}} \to \infty$ uniformly under bounded subsets $\mathcal{W}_2$ and uniformly under compact λ-intervals (see Fact 4.1), for $R_2 = R$ and $\epsilon = \min_{\lambda\in A} \frac{\|(Id-L(\lambda))^{-1}\|^{-1}_{\mathbb{H}^1_{2\pi}}}{4}$ there exists $R_1 > 0$ such that $P\nabla\Phi_V(u,\lambda) \ne 0$ for $\|u_1\|_{\mathbb{H}^1_{2\pi}} > R_1$ and $\|u_2\|_{\mathbb{H}^1_{2\pi}} \le R_2$. Since R_1, R_2 are chosen independent on $\lambda \in A = [\lambda_-,\lambda_0-\delta] \cup [\lambda_0+\delta,\lambda_+]$, we obtain

$$(\nabla\Phi_V)^{-1}(0) \cap ((\mathbb{H}^1_{2\pi} \setminus B_\varrho(\mathbb{H}^1_{2\pi})) \times [\lambda_-,\lambda_0-\delta] \cup [\lambda_0+\delta,\lambda_+]) = \emptyset,$$

where $\varrho > 0$ is sufficiently large radius such that $B_{R_1}(\mathbb{H}^1_{2\pi}) \times B_{R_2}(\mathbb{H}^1_{2\pi}) \subset B_\varrho(\mathbb{H}^1_{2\pi})$. $\square$

Remark 4.4. Notice that Theorem 4.5 is generalization of Theorem 4.3. of [18].

5. Examples

In this section we present two examples which illustrate the results proved in the previous sections.

Example 5.1. Let $x = (x_1,x_2) = (x_1^1,x_1^2,x_2^1,x_2^2) \in \mathbb{R}^4$. Consider system (3.1) with potential

$$V(x) = V(x_1,x_2) = \frac{1}{2}(Bx_1,x_1) + V_2(x_2) + W(x),$$

where

- $B = \begin{bmatrix} 2 & 0 \\ 0 & 6 \end{bmatrix}$,

- $V_2(x_2) = -\frac{1}{2}\left(\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} x_2, x_2\right)$,
- $W(x) = 8\ln(|x_2|^2 + 3)\mathrm{arctg}(|x_1|^2 + 1)$.

It is easy to check that potential V satisfies assumptions **A.1–A.5**. Denote $p_1 = (0, 0, 0, 0)$, $p_2 = (0, 0, \sqrt{4\pi - 3}, 0)$, $p_3 = (0, 0, -\sqrt{4\pi - 3}, 0)$, $p_4 = (0, 0, 0, \sqrt{\frac{4\pi-9}{3}})$, $p_5 = (0, 0, 0, -\sqrt{\frac{4\pi-9}{3}})$ and notice that $(V')^{-1}(0) = \{p_1, p_2, p_3, p_4, p_5\}$, thus assumption **A.6** is satisfied.

Note that $\sigma(V''(p)) \cap \{k^2 : k \in \mathbb{N} \cup \{0\}\} = \emptyset$ for every $p \in \{p_1, p_2, p_3, p_4, p_5\}$. Indeed:

$$V''(p_1) \approx \begin{bmatrix} 10,8 & 0 & 0 & 0 \\ 0 & 14,8 & 0 & 0 \\ 0 & 0 & 3,2 & 0 \\ 0 & 0 & 0 & 1,2 \end{bmatrix}$$

$$V''(p_2) = V''(p_3) \approx \begin{bmatrix} 22,2 & 0 & 0 & 0 \\ 0 & 26,2 & 0 & 0 \\ 0 & 0 & -1,5 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

$$V''(p_4) = V''(p_5) \approx \begin{bmatrix} 13,5 & 0 & 0 & 0 \\ 0 & 17,5 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -1,7 \end{bmatrix}.$$

Since $I_V(\infty, 2\pi)_k = (-1)^{j_0(B,2\pi)} \cdot j_k(B, 2\pi)$ and $I_V(p_i, 2\pi)_k = (-1)^{j_0(V''(p_i),2\pi)} \cdot j_k(V''(p_i), 2\pi)$ we obtain

$$I_V(\infty, 2\pi)_2 = 1 \neq 2 = 2 + 2 \cdot 2 + 2 \cdot (-2) = \sum_{i=1}^{5} I_V(p_i, 2\pi)_2.$$

Therefore all the assumptions of Theorem 4.1 are satisfied and there exist at least one 2π-periodic nonstationary solution of system (3.1).

Example 5.2. Let $x = (x_1, x_2) = (x_1^1, x_1^2, x_1^3, x_1^4, x_2^1, x_2^2) \in \mathbb{R}^6$. Consider system (4.2) with potential

$$V(x, \lambda) = V(x_1, x_2, \lambda) = \frac{1}{2}(B(\lambda)x_1, x_1) + V_2(x_2, \lambda) + W(x, \lambda),$$

where

- $B(\lambda) = \begin{bmatrix} 2 + \lambda\sqrt{2} & 0 & 0 & 0 \\ 0 & 7 + \lambda\sqrt{2} & 0 & 0 \\ 0 & 0 & -\lambda\sqrt{2} & 0 \\ 0 & 0 & 0 & -5 - \lambda\sqrt{2} \end{bmatrix}$,
- $V_2(x_2, \lambda) = -\lambda^2|x_2|^2(\ln(3 + |x_2|^2))^2$,
- $W(x, \lambda) = 8\lambda^2 \ln(|x_2|^2 + 3)\mathrm{arctg}(|x_1|^2 + 1)$.

Let $\lambda_- = 1 < \lambda_+ = 2$. Note that

(i) $\{\lambda \in [1, 2] : \sigma(B(\lambda)) \cap \{k^2 : k \in \mathbb{N} \cup \{0\}\} \neq \emptyset\} = \{\sqrt{2}\}$;
(ii) $(-V_2'(x_2, \lambda), x_2) = 2\lambda^2 \ln(3 + |x_2|^2) \cdot [\ln(3 + |x_2|^2) + 2\frac{|x_2|^2}{3+|x_2|^2}]|x_2|^2 \geq |x_2|^2$;
(iii) since

$$W'_{x_1}(x) = \frac{16\lambda^2 \ln(|x_2|^2 + 3)}{1 + (|x_1|^2 + 1)^2} \cdot x_1,$$

we obtain that $W'_{x_1}(x, \lambda_\pm) = o(|x_1|)$ as $|x_1| \to \infty$ uniformly under compact subsets of $\mathbb{R}^{n_2}$ and compact λ-intervals;
(iv) $(W'_{x_2}(x, \lambda), x_2) < 32\pi$ for every $x \in \mathbb{R}^6$;
(v) since

$$B(1) \approx \begin{bmatrix} 3,4 & 0 & 0 & 0 \\ 0 & 8,4 & 0 & 0 \\ 0 & 0 & -1,4 & 0 \\ 0 & 0 & 0 & -6,4 \end{bmatrix}, \qquad B(2) \approx \begin{bmatrix} 4,8 & 0 & 0 & 0 \\ 0 & 9,8 & 0 & 0 \\ 0 & 0 & -2,8 & 0 \\ 0 & 0 & 0 & -7,8 \end{bmatrix},$$

we obtain that

$$(-1)^{j_0(B(1),2\pi)} \cdot j_k(B(1), 2\pi) = 1 \cdot 1 = 1 \neq 2 = 1 \cdot 2 = (-1)^{j_0(B(2),2\pi)} \cdot j_k(B(2), 2\pi).$$

Since all the assumptions of Theorem 4.5 are satisfied, there is an unbounded closed connected component $C \subset \mathbb{H}^1_{2\pi} \times [1, 2]$ of solutions of (4.2) such that $C \cap (B_\gamma(\mathbb{H}^1_{2\pi}) \times \{1, 2\}) \neq \emptyset$. Moreover C meets $(\infty, \sqrt{2})$.

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